Chaotic thermovibrational flow in a laterally heated cavity

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Period-doubling transitions to chaos, periodic windows, strange attractors, and intermittencies are observed in direct numerical simulations of convection in a closed cavity with differentially heated vertical walls. The cavity contains a Newtonian-Boussinesq fluid and is subject to horizontal oscillatory displacements with a frequency Ω. The transitions occur through a sequence of bifurcations that exhibit the features of a Feigenbaum-type scenario. The first transition from a single-frequency response to a two-frequency response occurs through a parametric excitation of the subharmonic mode $\Omega/2$ by the driving frequency Ω. Bifurcation diagrams also exhibit periodic windows and reveal the self-similar structure of the ''period-doubling tree.'' Intermittent flows show characteristics corresponding to a Pomeau-Manneville type-I intermittency. [S1063-651X(97)03910-X]

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I. INTRODUCTION

Patterson and Imberger first reported an oscillatory approach to steady flow in a cavity subject to differential lateral heating [1]. The oscillations corresponded to damped internal gravity waves that arose when fluid from the vertical boundary layers intruded along the horizontal walls. This behavior was later explained as the result of internal hydraulic jumps [2]. Later work focused on the nature of flow transitions [3–7] and there has been some discussion on the mechanism responsible for the first transition to oscillatory flow. From the results of their numerical simulations, Chenoweth and Paolucci [3] suggested that hydraulic jumps were also responsible for the first transitions from steady to time-dependent flow. From the results of a detailed analysis of the corner flow structure, Ravi, Henckes, and Hoogendoorn [4] concluded that the flow did not have the character of an internal hydraulic jump, and that a thermal mechanism originating in the vertical boundary layers was responsible. In later work, Paolucci and Chenoweth [5] examined transitions to chaos in a differentially heated vertical cavity and found a second instability occurring in the vertical boundary layers that resulted in oscillations with frequencies that compared well to those predicted by Gill and Davey [6]. The question of the nature of the instability mechanisms responsible for the two instabilities was revisited by Janssen and Henckes [7]. They found a Prandtl number dependence for the transitions. For $0.25<\mathrm{Pr}<2.0$, they observed a transition from laminar to chaotic flow through intermediate periodic and quasiperiodic regimes as proposed by Ruelle and Takens [8]. At higher Prandtl numbers, no such intermediate regimes were found. They suggested that the transition from the steady to the periodic regime occurs through a shear-driven Kelvin-Helmholtz-type instability. The second transition, from periodic to quasiperiodic flow, appears to originate in the vertical boundary layers. Discrepancies between results obtained at $\mathrm{Pr}>0.71$ and the instability of the natural convection boundary layer at an isolated vertical plate suggest that the second transition is also shear driven. In previous work on transitions to chaotic flows in differentially heated cavities with adiabatic horizontal walls [5,7,9], all such transitions are characterized by large Rayleigh numbers ($\mathrm{Ra}>10^8$). In this work we show that when the cavity is subject to lateral oscillatory translations, we obtain transitions to chaos at $\mathrm{Ra}=10^4$.

Vertical oscillatory translations result in the appearance of an effective body force of the form $g=g_0+g_1 \sin \omega t$ where $g_1=b\omega^2$, ω is the angular frequency, and b corresponds to the displacement amplitude. The stability of a layer of fluid of infinite extent and heated from below, or above, is affected by gravity modulation (or vibration) and has received a limited amount of attention for the case of sinusoidally modulated gravity or vibration [10–14]. The oscillatory displacement of the rigid-walled container is accounted for by transforming the equations of motion to a frame of reference in which the container is stationary. In the well-known case for heating from below and with $\varepsilon=0$, the first bifurcation threshold occurs at a critical value of the Rayleigh number, Ra_c, and convective motion ensues. For nonzero g_1 the system can always be stabilized in some region of the b-ω plane for Rayleigh numbers in excess of Ra_c. This case is analogous to the case of an inverted rigid pendulum with an oscillating support point. On the other hand, the case of heating from above, for which the nonconvecting state is stable when $\varepsilon=0$, is destabilized for some range of b-ω with larger b and ω being the most destabilizing. For heating from below, the response of the flow as the vibration amplitude is increased evolves through synchronous to subharmonic and, ultimately, relaxation oscillations [10,13].

In recent experimental work on the effect of vibration on the Rayleigh-Bénard problem, Ishikawa and Kamei [15] observed quasiperiodic flows at 40 Ra_c. They simulated their experiments using a Lorenz model. As the frequency was increased at fixed modulation amplitude at values of Ra of about Ra_c, they found transitions from two-periodic to quasiperiodic and chaotic flows.

The effect of vibration on a convective motion has also been studied for the case of a differentially heated square cavity [16–18]. Farooq and Homsy [16] considered gravity modulation as a perturbation to steady gravity and found that for certain parametric conditions the periodic modulation interacts with instabilities associated with the base flow. This interaction produces resonances that increase in strength as Ra increases and also depends on the Prandtl number. The time-periodic forcing results in a streaming phenomenon where the flow can be separated into a mean part plus a time-dependent solution. The mean part of the flow is generally a combination of the flow that would persist at that Ra in the absence of gravity modulation and the mean flow generated by the periodic forcing. In full numerical simulations, Fu and Shieh [17] found that the strength of the resonant interactions is also dependent on the modulation amplitude. In later work, Farooq and Homsy [18] observed that parametric resonance in a gravity modulated differentially heated slot occurs in association with excitation of internal waves and leads to instability in the flow. The calculated stability boundary depends on the frequency and magnitude of the modulation amplitude. The minimum value of the modulation amplitude at which instability was found, in a remarkable analogy to the stability boundaries of the Mathieu equation, to correspond approximately to forcing frequencies of $2\Omega_R$, Ω_R, $2\Omega_R/5$, etc., where Ω_R corresponds to the fundamental resonant frequency of the system.

The mechanism for parametric resonance for the differentially heated slot appears to be a resonance between the forced oscillation of the basic flow and the free oscillations of stable perturbations of the time averaged flow. Farooq and Homsy [16,18] proposed that these oscillations are characterized by the Brünt-Väisäla frequency which represents the maximum possible frequency that can be supported by a *stable* oscillating stratified fluid [19].

In contrast to earlier studies, which involved modulation of the gravity vector, we examine the effect of oscillatory horizontal translations of a square cavity with differentially heated vertical walls using direct numerical simulations of the Navier-Stokes-Boussinesq equations. These simulations show that at high enough forcing amplitudes and certain frequencies a region of instability develops which leads to subharmonic cascades, intermittencies, and other chaoticlike behavior. In the absence of vibration, such behavior would only be expected at values of the Rayleigh number several orders of magnitude higher than the value considered here. Thus it appears that lateral vibration of the cavity leads to an early transition to chaos.

II. GOVERNING EQUATIONS

The cavity is subject to large amplitude horizontal accelerations of the form $b\omega^2 \sin(\omega t)\mathbf{i}$, where b and ω are the oscillation amplitude and frequency and $\mathbf{i}$ is a horizontal unit vector. The equations were formulated in a moving frame of reference in which the oscillatory translations appeared as a time-dependent body force in the momentum equations. Length, time, velocity, and temperature scales were taken to be L, L^2/κ, L/κ, and ΔT, where L, κ, and ΔT are the cavity length, the thermal diffusivity, and the horizontal temperature differential, respectively. This leads to the dimensionless body forces

$$\mathrm{Pr\,Ra}\; T(\mathbf{x},t)\mathbf{j} + F\,\sin(\Omega t)T(\mathbf{x},t)\mathbf{i}. \tag{1}$$

Here $T(\mathbf{x},t)$ is the temperature at a point $\mathbf{x}$ at time t, and $\mathbf{j}$, Ω, and $\mathrm{Ra}=g\beta\Delta T L^3/\upsilon\kappa$ are the vertical unit vector, dimensionless frequency, and Rayleigh number, respectively, and $F=\Omega\sqrt{2\mathrm{Ra}_v\mathrm{Pr}}$, where $\mathrm{Ra}_v=(b\omega\beta\Delta TL)^2/2\upsilon\kappa$ is the vibrational Rayleigh [11] number and Pr is the Prandtl number. The product $(\mathrm{Ra}_v\mathrm{Pr})^{1/2}$ represents the ratio of the characteristic times for heat diffusion and average fluid motion due to vibration. The equations governing the transport of mass, momentum, and heat take the form

$$\boldsymbol{\nabla}\cdot\mathbf{u}=0, \tag{2}$$

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial\mathbf{u}}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right)=-\boldsymbol{\nabla}p+\nabla^2\mathbf{u}+\mathrm{Ra}\;T\mathbf{k} + \Omega\sqrt{2\mathrm{Ra}_v/\mathrm{Pr}}\,\sin(\Omega t)T\mathbf{i}, \tag{3}$$

$$\frac{\partial T}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}T=\nabla^2 T. \tag{4}$$

Here $\mathbf{u}=\mathbf{u}(\mathbf{x},t)$ and p, are, respectively, the velocity and pressure. The velocity vanishes on the rigid cavity walls.

III. SOLUTION METHOD

The governing equations were solved in a vorticity-streamfunction form using a pseudospectral Chebyshev collocation method [20–22]. Time discretization is achieved using an Adams-Bashforth–second-order backward Euler scheme [20,22] while the spatial dependence of the vorticity and pressure is approximated using collocated Chebyshev polynomials. The problem of vorticity boundary conditions was surmounted by employing an ‘‘influence matrix method’’ described in detail in Ref. [21]. Selected (nontrivial) cases were repeated using the spectral element code NEKTON [23] to assess the fidelity of the results. NEKTON employs an implicit scheme for viscous and diffusive terms and an explicit third-order scheme for inertial and source terms. All cases compared were found to be in good agreement.

IV. RESULTS

Our results for $\mathrm{Ra}=10^4$ and $\mathrm{Pr}=0.71$ are summarized in Fig. 1. Observed flow regimes included a quasisteady (I) regime, where inertial effects are negligible, an oscillatory regime (II) where the velocity is out of phase with the driving force, and an asymptotic regime (III) where inertial effects dominate and the flow oscillates with small amplitude about the mean flow. A fourth regime, which first appears at $F\geqslant F^*=4.36\times10^4$ for $106<\Omega<112$, is characterized by subharmonic cascades and chaotic behavior. The edge of a second region of instability was also detected at larger F values for $\Omega\approx87$.

For $F<F^*$, the response in regimes I–III is characterized by single-frequency oscillations. For $F>F^*$ there is a band of frequencies for which marked transitions in the flow are observed. This band increases in width as Ra_v (and thus F) increases. For a given frequency in this band, the region of instability is contained between two values of F. As the region is entered from below, there is a sequence of subharmonic bifurcations leading to chaos which exhibit properties characteristic of a Feigenbaum-type scenario. At the upper limit of the region the behavior is typically characterized by

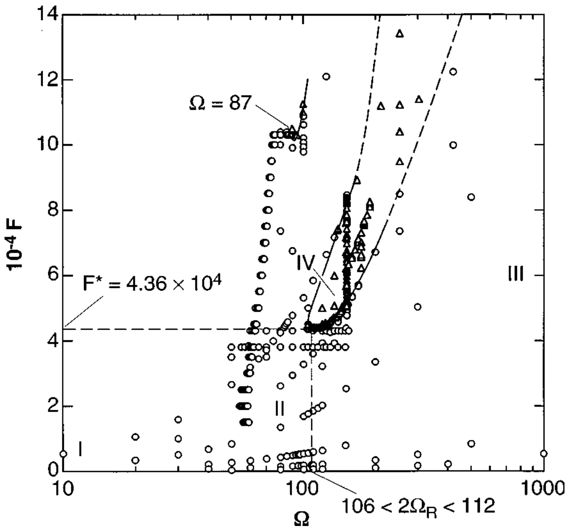

FIG. 1. Flow regimes for the $Ra = 10^4$, $Pr = 0.71$ case as a function of $F = \Omega\sqrt{2 Ra_v Pr}$ and Ω. The solid line is an estimate of the location of the boundary of the unstable region. Note the edge of a second unstable region at $\Omega = 87$ for $F \sim 10^5$. (All cases shown as Δ lie within regions of instability.) I—quasisteady regime, II—oscillatory regime, III—asymptotic regime.

intermittencies. Upon exiting the unstable region the flow is again monoperiodic and has a frequency equal to the forcing frequency.

Figure 2 is a bifurcation diagram obtained from Poincaré sections of a vertical velocity component for $\Omega = 152$ and different values of Ra_v. For $Ra_v < 3.94 \times 10^4$ the velocity exhibits a single-frequency response. At higher Ra_v there is a subharmonic cascade. Within the subharmonic cascade we found the locations, s_k, of the superattractive points between the $\Omega/2$, $\Omega/4$, $\Omega/8$, and $\Omega/16$ bifurcations (here $k = 1,2,3,...$ corresponds to periodic cycles with $\Omega/2$, $\Omega/4$, $\Omega/8$, etc.). We then computed the values $\delta_k = (s_k - s_{k-1})/(s_{k-1} - s_{k-2})$ of the first two iterates of the Feigenbaum sequence. We found $\delta_1 = 4.9491$ and $\delta_2 = 4.9104$. This compares well with the values 4.4 ± 0.1 obtained in experiments with mercury [24] and with the theoretical value $\delta_\infty = 4.6692$ as $k \to \infty$ [25].

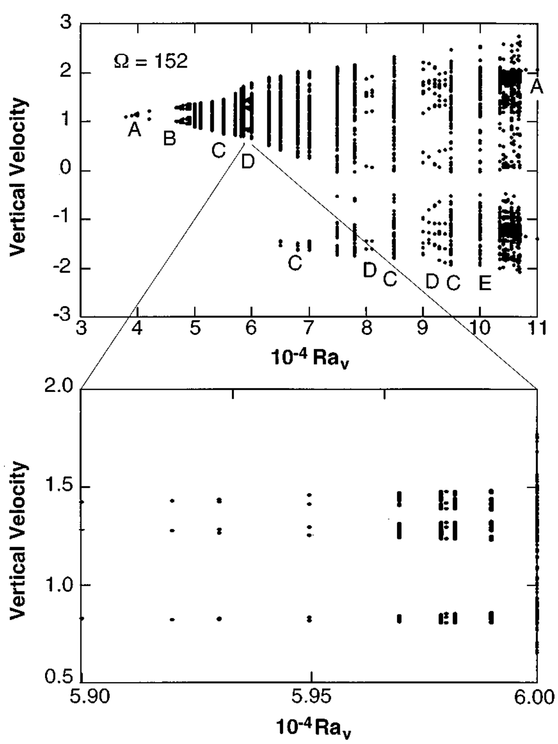

FIG. 2. (a) Bifurcation diagram showing vertical velocity values from a fixed location taken from Poincaré sections for different values of Ra_v and $\Omega = 152$; (b) detail for $5.9 < 10^{-4}\ Ra_v < 6.0$. *A*—one-frequency response; *B*—subharmonic cascade; *C*—strange attractor; *D*—periodic window; *E*—intermittency.

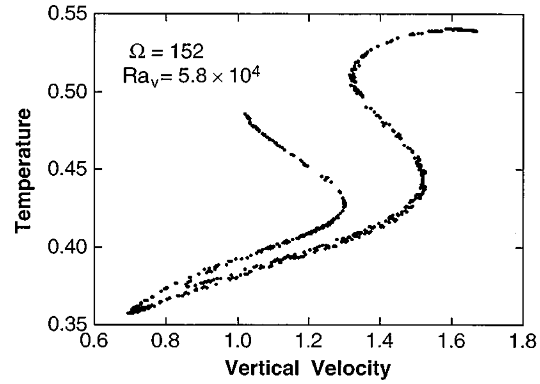

FIG. 3. Poincaré section for the temperature at a point in the cavity for $Ra_v = 58\,000$, $\Omega = 152$.

In the regions marked C, the characteristics of the response suggest the existence of a strange attractor. To investigate this further we followed the phase-space trajectories computed for $Ra_v = 5.8 \times 10^4$ and determined the existence of a positive Lyapunov exponent. In addition, inspection of the Poincaré sections (e.g., Fig. 3) for this case shows points that condense to form a well-defined pattern. This implies the existence of a negative Lyapunov exponent [26]. A positive Lyapunov exponent is the signature of a chaotic state while a negative Lyapunov exponent causes a contraction of the attractor in phase space. Their simultaneous existence is characteristic of a strange attractor [27]. An estimate of the attractor's fractal dimension, the box counting dimension D_b [27], was computed at two different locations. We found $D_b = 2.11$ and 2.2. The computation used to generate the data from which D_b was computed was carried out for 300 000 time steps corresponding to 750 periods of the driving frequency.

In the interval $5.9 \times 10^4 < Ra_v < 6 \times 10^4$ we found a period-3 window [shown in detail in Fig. 2(b)]. Each of the three points within the window undergoes a sequence of period-doubling bifurcations and a reverse cascade (note especially the period-3 window on each branch). This mimics the behavior of the bifurcation diagram on a larger scale and reflects its self-similar nature. As Ra_v is further increased there are more periodic windows (D) and chaotic regimes (C). Finally, at $Ra_v = 10^5$, type-I intermittencies [27] (E) were found. Theoretically, the number N of "periodic or laminar phases" between intermittent bursts should be proportional to $\varepsilon^{-1/2}$ [28]. Here $\varepsilon = Ra_v - Ra_{v,c}$ and the transition from oscillatory to intermittent flow occurs at $Ra_{v,c}$. We examined this as follows. First we obtained d, the shortest distance between the closest point of a first return map (see

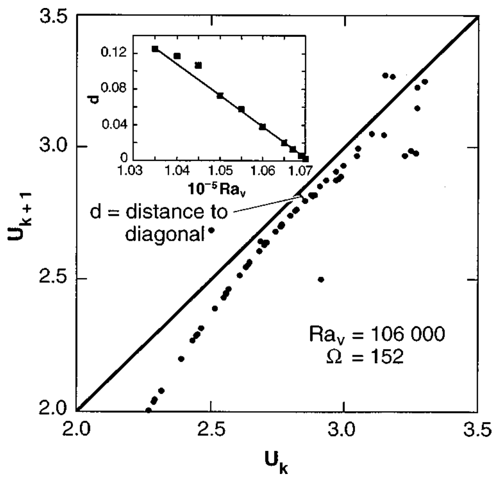

FIG. 4. First return map for a velocity component at a fixed location. Inset shows the dependence of d on Ra_v.

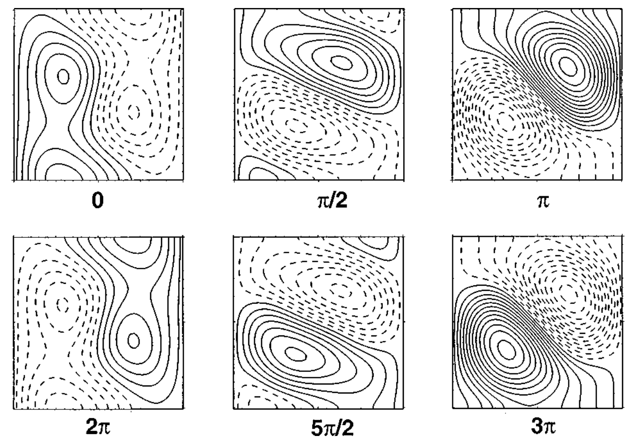

FIG. 6. The temperature mode $T_{\Omega/2}(\mathbf{x})\cos[\frac{1}{2}\Omega t+\phi_{\Omega/2}(\mathbf{x})]$ at $\Omega t=0$, $\pi/2$, π, 2π, $5\pi/2$, and $3\pi/2$ for the temperature at $\mathrm{Ra}_v=42\,000$. The disturbance rotates around the cavity in a clockwise sense. The left-hand wall is the hot wall.

Fig. 4) to the diagonal $u_{k+1}=u_k$, for different values of Ra_v. $\mathrm{Ra}_{v,c}$ was then obtained by linear regression (see inset in Fig. 4). As d approaches zero, N approaches infinity, i.e., there is a transition from an intermittent to a periodic response. For each Ra_v, we determined N, and found that $N \propto \varepsilon^{-0.53}$.

We isolated the flow and temperature modes using Fourier time-series analysis at each spatial location. Figures 5 and 6 show, respectively, the thermal Ω and $\Omega/2$ modes for the case $\mathrm{Ra}_v=4.2\times10^4$ and $\Omega=152$, i.e., inside the region of instability shown in Fig. 2. The Ω mode consists of two waves that travel toward the cavity center from the upper right (cold) and lower left (hot) corners. The cavity center is a node and the diagonal connecting the upper hot corner with the lower cold corner is roughly a nodal line. The thermal $\Omega/2$ mode, in contrast, rotates around the cavity in a clockwise sense. As Ra_v is increased above 3.94×10^4, perturbations to the thermal Ω mode lead to the transfer of hot and cold packets across the Ω-mode nodal line. These packets rotate around the cavity requiring two periods of the Ω-mode oscillation to complete their cycle. However, when the cavity is actually driven at $\Omega=76$, we found that the thermal Ω mode appears as a standing wave with an approximately vertical nodal line which cuts the cavity in half. We conjecture that, at the first period-doubling transition, this lower frequency mode is parametrically excited and the resulting nonlinear interaction with the Ω mode leads to a disturbance that travels around the cavity.

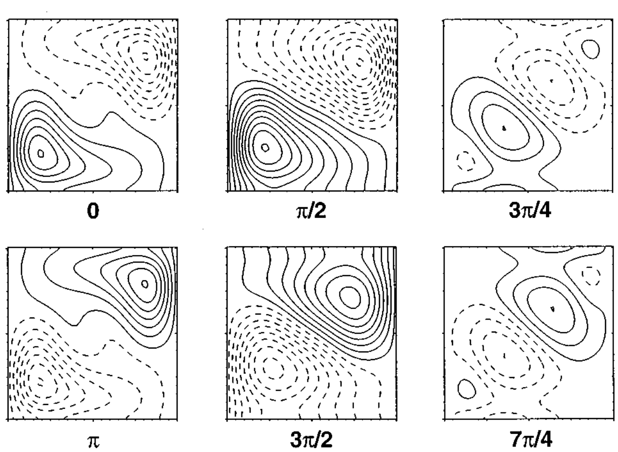

FIG. 5. The temperature mode $T_\Omega(\mathbf{x})\cos[\Omega t+\phi_\Omega(\mathbf{x})]$ at $\Omega t=0$, $\pi/2$, $3\pi/4$, π, $3\pi/2$, and $7\pi/4$ for $\mathrm{Ra}_v=42\,000$. A traveling wave moves from the corners toward the center. The left-hand wall is the hot wall.

V. DISCUSSION

In summary, we have observed transitions to chaotic flow in large amplitude thermovibrational convection in a laterally heated square cavity subject to vertical gravity. The initial period-doubling transition appears to be due to a parametric excitation. The calculated period-doubling route to chaos is characteristic of a Feigenbaum-type scenario. Such routes to chaos have been observed experimentally [28,29] and in hydrodynamic models (for example, [15,30–32]). In the absence of the type of sinusoidal buoyant forcing examined in this paper, transitions to chaotic flow in the differentially heated cavity would be expected to take place at Rayleigh numbers in excess of 10^8. There are only a few examples of direct numerical solutions of the Navier-Stokes equations that have examined transitions to chaos in differentially heated cavities [5,8,9]. These works have focused more on determining the mechanisms for steady to periodic and periodic to quasiperiodic transitions and do not attempt a detailed characterization of the path toward chaotic flow. It has been shown [33] that, for discrete-time approximations to the Navier-Stokes equations, one should observe the universally scaled accumulation of period-doubling bifurcations. However, we are unaware of any previous results based on direct numerical simulations of flows in differentially heated cavities that have been able to capture the details of a period-doubling bifurcation sequence including period-3 windows and clear evidence of a self-similar structure to the bifurcation. We found that, in the presence of sinusoidal forcing, early transitions (that is, at relatively low values of Ra) to chaos occur through a sequence of bifurcations that exhibit the features of a Feigenbaum-type scenario. The first transition from a single-frequency response to a two-frequency response occurs through a parametric excitation of the subharmonic mode $\Omega/2$ by the driving frequency Ω. In addition to the period-doubling bifurcation sequences we also found intermittent behavior near the upper boundary of the unstable region. These flows show characteristics corresponding to a Pomeau-Manneville type-I intermittency. While it might be argued that the discretization methods may be responsible for the observed behavior we emphasize that two entirely different methods were used to obtain these results. We also emphasize that these flows cannot be considered in terms of a small perturbation superimposed on a mean flow that is close to the $\mathrm{Ra}_v = 0$ flow. Indeed, the mean flow and temperature fields are quite different from the $\mathrm{Ra}_v = 0$ fields and the oscillation amplitudes too large to neglect nonlinearities. The results of this work and the accessibility of such flows to computational simulation reveal the richness of thermovibrational flows as a fruitful area of research on nonlinear fluid dynamics.

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